Another property of minimal surfaces in E^3

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Abstract

The new property of minimal surfaces is obtained in this article. We derive the equations of $\Delta-$recurrent and $\Delta-$harmonic surfaces in E^3 and prove that each minimal surface is $\Delta-$recurrent one with eigenvalue $\varphi = 2K$, where K is Gaussian curvature of surface. We conclude that each minimal surface in E^3 satisfies the equation $\Delta b = 2Kb$ and obtain that each $\Delta-$harmonic minimal surface is a plane $E^2 \subset E^3$ or its part.

Introduction

Let F^2 be a smooth two-dimensional surface in the three-dimensional Euclidean space E^3, g be the induced Riemmanian metric on F^2, ∇ be the Riemmanian connection on F^2, determined by g, b be the second fundamental form, $\overline{\nabla}$ be the Van der Varden – Bortolotti covariant derivative, Δ be the Laplas operator.

Definition 1 . *The second fundamental form b is called harmonic if $\Delta b \equiv 0$ on F^2.*

Theorem 1 . *If minimal surface F^2 in E^3 has harmonic second fundamental form then F^2 is a plane $E^2 \subset E^3$ or its part.*

Example 1. Straight round cylinder F^2 is a $\Delta-$harmonic surface in E^3 (see [1]), i.e. $\Delta b \equiv 0$ on F^2.

Definition 2 . *Surface F^2 in E^3 is called $\Delta-$recurrent with eigenvalue φ, if the function φ on F^2 satisfies the condition*

$$\Delta b = \varphi b.$$

Definition 3 . *Surface F^2 in E^3 is called $\Delta-$harmonic, if the second fundamental form b satisfies the condition*

$$\Delta b = 0.$$

Theorem 2 . *Each minimal surface F^2 in E^3 is $\Delta-$recurrent with eigenvalue $\varphi = 2K$, where K is the Gaussian curvature of surface.*

1 Equations of $\Delta-$recurrent and $\Delta-$harmonic surfaces in E^3

Let x be an arbitrary point of F^2, (x^1, x^2, x^3) be the Cartesian coordinates in E^3, (u^1, u^2) be the local coordinates on F^2 in some neighborhood $U(x)$ of the point x. Then F^2 is given locally by the vector equation

$$\vec{r} = \vec{r}(u^1, u^2) = \{x^1(u^1, u^2), x^2(u^1, u^2), x^3(u^1, u^2)\},$$

where

$$\mathrm{rg} \left\| \frac{\partial x^a}{\partial u^i} \right\| = 2 \quad, \forall y \in U(x).$$

Let us fix a point $x \in F^2$ and introduce the isothermal coordinates (u^1, u^2) in some nighborhood $U(x)$ on F^2. Then the induced metric is $g = A(u^1, u^2)((du^1)^2 + (du^2)^2)$.

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Then

$$g^{11} = g^{22} = \frac{1}{A}, \quad g^{12} = 0.$$

We derive the Christoffel symbols:

$$\Gamma^1_{11} = \Gamma^2_{12} = -\Gamma^1_{22} = \frac{1}{2A}\partial_1 A, \quad \Gamma^1_{12} = \Gamma^2_{22} = -\Gamma^2_{11} = \frac{1}{2A}\partial_2 A.$$

Putting $B = (\ln A)/2$, we have:

$$\Gamma^1_{11} = \Gamma^2_{12} = -\Gamma^1_{22} = \partial_1 B, \quad \Gamma^1_{12} = \Gamma^2_{22} = -\Gamma^2_{11} = \partial_2 B.$$

We denote by $\{\vec{n}\}$ the field of unit normal vectors in the normal bundle $T^{\perp}F^2$ in $U(x)$: $\vec{n} = \vec{n}(u^1, u^2)$, $< \vec{n}, \vec{n} >= 1$, where $<,>$ is scalar product in E^3. Let $b_{ij} =< \partial^2_{ij}\vec{r}, \vec{n} >$ be the coefficients of the second fundamental form $b =< d^2\vec{r}, \vec{n} >= b_{ij}du^i du^j$. The covariant derivatives $\nabla_i b_{jk}$ are

$$\nabla_i b_{jk} = \partial_i b_{jk} - \Gamma^m_{ij} b_{mk} - \Gamma^m_{ik} b_{jm}.$$

The Laplas operator components Δb in $U(x)$ are (see [2]):

$$(\Delta b)_{ij} = g^{kl}\overline{\nabla}_k \overline{\nabla}_l b_{ij}.$$

Since the normal connection of F^2 in E^3 is planar, we have in coordinates (u^1, u^2):

$$A(\Delta b)_{ij} = \nabla_1 \nabla_1 b_{ij} + \nabla_2 \nabla_2 b_{ij}.$$

We obtain $\nabla_i b_{jk}$:

$$\nabla_1 b_{11} = \partial_1 b_{11} - 2b_{11}\partial_1 B + 2b_{12}\partial_2 B,$$
$$\nabla_1 b_{12} = \partial_1 b_{12} - 2b_{12}\partial_1 B + b_{22}\partial_2 B - b_{11}\partial_2 B,$$
$$\nabla_1 b_{22} = \partial_1 b_{22} - 2b_{12}\partial_2 B - 2b_{22}\partial_1 B,$$
$$\nabla_2 b_{11} = \partial_2 b_{11} - 2b_{11}\partial_2 B - 2b_{12}\partial_1 B,$$
$$\nabla_2 b_{12} = \partial_2 b_{12} - 2b_{12}\partial_2 B - b_{22}\partial_1 B + b_{11}\partial_1 B,$$
$$\nabla_2 b_{22} = \partial_2 b_{22} + 2b_{12}\partial_1 B - 2b_{22}\partial_2 B.$$

The Gauss equation is:

$$b_{11}b_{22} - b^2_{12} = -\frac{A}{2}\Delta \ln A. \tag{1}$$

The Peterson-Codacci equation system, written in coordinates (u^1, u^2), is:

$$\partial_2 b_{11} - \partial_1 b_{12} = (b_{11} + b_{22})\partial_2 B, \tag{2}$$

$$\partial_1 b_{22} - \partial_2 b_{12} = (b_{11} + b_{22})\partial_1 B. \tag{3}$$

Let us denote $\Delta^e b_{ij} = \partial^2_{11} b_{ij} + \partial^2_{22} b_{ij}$, $\Delta B = \partial^2_{11} B + \partial^2_{22} B$. Using the equations (2) and (3) we have:

$$A(\Delta b)_{11} = \Delta^e b_{11} - 2b_{11}\Delta B - 4(\partial_1 b_{11} + \partial_1 b_{22})\partial_1 B + 2(b_{11} + b_{22})(3(\partial_1 B)^2 - (\partial_2 B)^2),$$

$$A(\Delta b)_{12} = \Delta^e b_{12} - 2b_{12}\Delta B - 2(\partial_1 b_{11} + \partial_1 b_{22})\partial_2 B - 2(\partial_2 b_{11} + \partial_2 b_{22})\partial_1 B +$$
$$+8(b_{11} + b_{22})\partial_1 B \partial_2 B,$$

$$A(\Delta b)_{22} = \Delta^e b_{22} - 2b_{22}\Delta B - 4(\partial_2 b_{11} + \partial_2 b_{22})\partial_2 B + 2(b_{11} + b_{22})(3(\partial_2 B)^2 - (\partial_1 B)^2).$$

Putting $u = b_{11} + b_{22}$, we obtain:

$$A(\Delta b)_{11} = \Delta^e b_{11} - 2b_{11}\Delta B - 4\partial_1 B \partial_1 u + 2u(3(\partial_1 B)^2 - (\partial_2 B)^2),$$

$$A(\Delta b)_{12} = \Delta^e b_{12} - 2b_{12}\Delta B - 2\partial_1 B\partial_2 u - 2\partial_2 B\partial_1 u + 8u\partial_1 B\partial_2 B,$$
$$A(\Delta b)_{22} = \Delta^e b_{22} - 2b_{22}\Delta B - 4\partial_2 B\partial_2 u + 2u(3(\partial_2 B)^2 - (\partial_1 B)^2). \quad (4)$$

The surface F^2 in E^3, by the definition, is $\Delta-$recurrent with eigenvalue φ if the function φ satisfies the condition

$$(\Delta b)_{ij} = \varphi b_{ij}, \quad i,j = 1,2,$$

which is equal to the following equation system:

$$\Delta^e b_{11} - 2b_{11}\Delta B - 4\partial_1 B\partial_1 u + 2u(3(\partial_1 B)^2 - (\partial_2 B)^2) = A\varphi b_{11},$$
$$\Delta^e b_{12} - 2b_{12}\Delta B - 2\partial_1 B\partial_2 u - 2\partial_2 B\partial_1 u + 8u\partial_1 B\partial_2 B = A\varphi b_{12},$$
$$\Delta^e b_{22} - 2b_{22}\Delta B - 4\partial_2 B\partial_2 u + 2u(3(\partial_2 B)^2 - (\partial_1 B)^2) = A\varphi b_{22}. \quad (5)$$

The equation system (1) — (3), (5) determines $\Delta-$recurrent surfaces F^2 in E^3.

The equality $\Delta b \equiv 0$ on F^2 in E^3 is equivalent to the following equation system:

$$1) \quad \Delta^e b_{11} - 2b_{11}\Delta B - 4\partial_1 B\partial_1 u + 2u(3(\partial_1 B)^2 - (\partial_2 B)^2) = 0,$$
$$2) \quad \Delta^e b_{12} - 2b_{12}\Delta B - 2\partial_1 B\partial_2 u - 2\partial_2 B\partial_1 u + 8u\partial_1 B\partial_2 B = 0,$$
$$3) \quad \Delta^e b_{22} - 2b_{22}\Delta B - 4\partial_2 B\partial_2 u + 2u(3(\partial_2 B)^2 - (\partial_1 B)^2) = 0. \quad (6)$$

The equation system (1) — (3), (6) determines $\Delta-$harmonic surfaces F^2 in E^3.

2 $\Delta-$harmonic minimal surfaces

Proof of the theorem 1. Using the condition on the mean curvature $H \equiv 0$, we have $u = b_{11} + b_{22} = 0$. Then the equation system (2) and (3) is:

$$\partial_1 b_{12} = \partial_2 b_{11}, \quad \partial_2 b_{12} = \partial_1 b_{22}.$$

Therefore,

$$\partial^2_{22} b_{11} = \partial^2_{11} b_{22}, \quad \partial^2_{11} b_{12} + \partial^2_{22} b_{12} = \partial^2_{21} b_{11} + \partial^2_{12} b_{22}. \quad (7)$$

Using (7), we derive:

$$\Delta^e b_{11} = \partial^2_{11} b_{11} + \partial^2_{22} b_{11} = \partial^2_{11} b_{11} + \partial^2_{11} b_{22} = \partial^2_{11} u = 0,$$
$$\Delta^e b_{12} = \partial^2_{11} b_{12} + \partial^2_{22} b_{12} = \partial^2_{21} b_{11} + \partial^2_{12} b_{22} = \partial^2_{12} u = 0,$$
$$\Delta^e b_{22} = \partial^2_{11} b_{22} + \partial^2_{22} b_{22} = \partial^2_{22} b_{11} + \partial^2_{22} b_{22} = \partial^2_{22} u = 0. \quad (8)$$

Using (8), we obtain from the equation system (6):

$$b_{11}\Delta B = 0, \quad b_{12}\Delta B = 0, \quad b_{22}\Delta B = 0.$$

Hence,

$$(b_{11}b_{22} - b^2_{12})(\Delta B)^2 = 0. \quad (9)$$

Using the equation (1), we obtain from the equation (9):

$$\left(-\frac{A}{2}\Delta \ln A\right)(\Delta B)^2 = 0.$$

Consequently, $(\Delta B)^3 = 0$. Hence, $\Delta B = 0$ in $U(x)$ and

$$K = -\frac{1}{2A}\Delta(\ln A) = -\frac{\Delta B}{A} = 0.$$

Therefore, the equations

$$K = 0, \quad H = 0$$

are valid in $U(x)$ on F^2.

Hence, $U(x)$ is an open part of a plane $E^2 \subset E^3$.

The theorem 1 is proved.

3 $\Delta-$recurrence of the second fundamental form of minimal surfaces

Proof of the theorem 2. From the condition $u = b_{11} + b_{22} = 0$ we have, that the equation system (8) is valid in $U(x)$. Using (8), we have from (4):

$$A(\Delta b)_{11} = -2b_{11}\Delta B, \quad A(\Delta b)_{12} = -2b_{12}\Delta B, \quad A(\Delta b)_{22} = -2b_{22}\Delta B.$$

Therefore, observing $-\Delta B = AK$, we derive:

$$(\Delta b)_{11} = 2Kb_{11}, \quad (\Delta b)_{12} = 2Kb_{12}, \quad (\Delta b)_{22} = 2Kb_{22}.$$

Let us put $\varphi = 2K$.

Therefore, the equation $\Delta b = \varphi b$, where $\varphi = 2K$, is valid in $U(x) \subset F^2$.

The theorem 2 is proved.

Example 2. Let the surface F^2 is given locally by the vector equation:

$\vec{r} = \{x^1 - \frac{4}{3}(x^1)^3 + 4(x^1)(x^2)^2, x^2 - \frac{4}{3}(x^2)^3 + 4(x^1)^2(x^2), 2(x^1)^2 - 2(x^2)^2\}$

Hence, $g_{11} = (4(x^1)^2 + 4(x^2)^2 + 1)^2$, $g_{22} = g_{11}, g_{12} = g_{21} = 0$,
$g^{11} = \frac{1}{g_{11}}, g^{22} = g^{11}, g^{12} = g^{21} = 0.$

The unit normal vector:
$\vec{n} = \{\frac{-16(x^1)(x^2)^2 - 4(x^1) - 16(x^1)^3}{g_{11}}, \frac{4(x^2) + 16(x^1)^2(x^2) + 16(x^2)^3}{g_{11}}, \frac{1 - 32(x^1)^2(x^2)^2 - 16(x^1)^4 - 16(x^2)^4}{g_{11}}\}$

$b_{11} = 4, b_{22} = -4, b_{12} = b_{21} = 0.$

Therefore, the mean curvature $H = 0$ and the Gaussian curvature $K = \frac{-16}{(g_{11})^2}$.

The covariant derivatives:
$\nabla_1 b_{11} = -\frac{64(x^1)}{\sqrt{g_{11}}}$, $\nabla_2 b_{12} = \nabla_2 b_{21} = \nabla_1 b_{22} = -\nabla_1 b_{11}$
$\nabla_2 b_{11} = -\frac{64(x^2)}{\sqrt{g_{11}}}$, $\nabla_1 b_{21} = \nabla_1 b_{12} = -\nabla_2 b_{22} = \nabla_2 b_{11}$

$\nabla_2\nabla_1 b_{11} = \frac{3584(x^1)(x^2)}{g_{11}}$,
$\nabla_1\nabla_2 b_{11} = \nabla_1\nabla_1 b_{12} = \nabla_2\nabla_2 b_{12} = \nabla_1\nabla_1 b_{21} =$
$-\nabla_2\nabla_2 b_{21} = -\nabla_2\nabla_1 b_{22} = -\nabla_1\nabla_2 b_{22} = \nabla_2\nabla_1 b_{11}.$

$\nabla_1\nabla_1 b_{11} = \frac{64(-28(x^2)^2 - 1 + 28(x^1)^2)}{g_{11}}$,
$\nabla_1\nabla_2 b_{12} = \nabla_1\nabla_2 b_{21} = \nabla_1\nabla_1 b_{22} = -\nabla_1\nabla_1 b_{11}.$

$\nabla_2\nabla_2 b_{11} = -\frac{64(-28(x^2)^2 + 1 + 28(x^1)^2)}{g_{11}}$,
$\nabla_2\nabla_1 b_{12} = \nabla_2\nabla_1 b_{21} = -\nabla_2\nabla_2 b_{22} = \nabla_2\nabla_2 b_{11}.$

$(\Delta b)_{11} = -\frac{128}{(g_{11})^2}, (\Delta b)_{22} = -(\Delta b)_{11}, (\Delta b)_{12} = 0, (\Delta b)_{21} = 0.$

Therefore, we have $\Delta b = 2Kb$.

Example 3. Let the surface F^2 is given locally by the vector equation:

$\vec{r} = \{x^2\cos(x^1), x^2\sin(x^1), x^1\}$

$g_{11} = 1 + (x^2)^2, g_{22} = 1, g_{12} = g_{21} = 0.$

$g^{11} = \frac{1}{1+(x^2)^2}, g^{22} = 1, g^{12} = g^{21} = 0.$

The unit normal vector:
$\vec{n} = \{-\frac{\sin(x^1)}{\sqrt{1+(x^2)^2}}, \frac{\cos(x^1)}{\sqrt{1+(x^2)^2}}, \frac{-x^2}{\sqrt{1+(x^2)^2}}\}$

$b_{11} = b_{22} = 0, b_{12} = b_{21} = \frac{1}{\sqrt{1+(x^2)^2}}$.

Therefore, the mean curvature $H = 0$ and the Gaussian curvature $K = -\frac{1}{(1+(x^2)^2)^2}$.

The covariant derivatives:

$\nabla_1 b_{11} = \frac{2(x^2)}{\sqrt{1+(x^2)^2}}$,

$\nabla_2 b_{11} = \nabla_1 b_{12} = \nabla_1 b_{21} = \nabla_2 b_{22} = 0$

$\nabla_2 b_{12} = \nabla_2 b_{21} = \nabla_1 b_{22} = \frac{-2x^2}{\sqrt{(1+(x^2)^2)^3}}$.

$\nabla_1\nabla_1 b_{11} = \nabla_2\nabla_2 b_{11} = \nabla_2\nabla_1 b_{12} = \nabla_1\nabla_2 b_{12} = \nabla_2\nabla_1 b_{21} = \nabla_1\nabla_2 b_{21} = \nabla_1\nabla_1 b_{22} = \nabla_2\nabla_2 b_{22} = 0.$

$\nabla_1\nabla_2 b_{11} = \nabla_1\nabla_1 b_{12} = \nabla_1\nabla_1 b_{21} = -\nabla_1\nabla_2 b_{22} = -\frac{6(x^2)^2}{\sqrt{(1+(x^2)^2)^3}}$.

$\nabla_2\nabla_1 b_{11} = -\frac{2(-1+3(x^2)^2)}{\sqrt{(1+(x^2)^2)^3}}$

$\nabla_2\nabla_2 b_{12} = \nabla_2\nabla_2 b_{21} = \nabla_2\nabla_1 b_{22} = \frac{2(-1+3(x^2)^2)}{\sqrt{(1+(x^2)^2)^5}}$.

$(\Delta b)_{12} = -\frac{2}{\sqrt{(1+(x^2)^2)^5}}, (\Delta b)_{21} = (\Delta b)_{12}, (\Delta b)_{11} = 0, (\Delta b)_{22} = 0.$

Therefore, we have $\Delta b = 2Kb$.

References

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